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On the Simultaneous Verifiability of Yes–No Measurements

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The relation of simultaneous verifiability (compatibility) of yes–no measurements, introduced by G. W. Mackey for the purposes of quantum axiomatics, is investigated. The meaning of this important relation is clarified here by showing its position among all the so-called weak compatibilities defined axiomatically in the logic of propositions.

1. PRELIMINARIES, DEFINITIONS, AND NOTATION

There are three basic concepts at the foundations of any physical theory: states, observables, and yes–no measurements (called also propositions, questions, events), i.e., measurements that can give only two results, yes or no. A common belief of physicists is that every measurement of a physical quantity can be reduced to a series of yes–no measurements, and thus we may expect the structure of the set of all observables to be completely determined by the structure of the set of all yes–no measurements. This is indeed so [see, e.g., Mackey (1963)], and therefore the structure of the set of propositions (which we call the *logic of a physical system*; briefly, a *logic*) is of primary importance.

The logic of any classical system is found to be the Boolean lattice of all Borel subsets of the phase space of the system, while for a quantum mechanical system its logic is the complete ortholattice of all closed subspaces of a (separable complex) Hilbert space corresponding to the system.

For an abstract logic L of a general physical system (without considering it concretely represented) the following properties are assumed to hold (Mackey, 1963):

(L1) L is a *σ-orthoposet*, that is, a σ-orthocomplete orthocomplemented partially ordered set.

(L2) L is *orthomodular*, i.e., $a \leqslant b$ ($a, b \in L$) implies $b = a \vee c$ for some $c \in L$, $c \perp a$. (We write $c \perp a$ and say that propositions c and a are *orthogonal*, if $c \leqslant a'$ or, equivalently, $a \leqslant c'$, where the prime denotes orthocomplementation in L. This orthogonality relation is obviously symmetric).

Having the logic L fixed, we can identify the states of the physical system with probability measures on L and the observables with σ-homomorphisms from the Borel subsets of the real line R^1 into L [see, e.g., Mackey (1963)].

2. A CHARACTERIZATION OF THE SIMULTANEOUS VERIFIABILITY OF PROPOSITIONS

Usually the most important manifestation of the quantum nature of microphenomena is considered the existence of such observables, which do not admit a simultaneous measurement with an arbitrary accuracy. It can be shown that this is exactly equivalent to the existence of propositions that are not simultaneously verifiable [for a precise proof, see Varadarajan (1962)], where, by definition, propositions $a, b \in L$ are said to be *simultaneously verifiable* [or *compatible*, $a \leftrightarrow b$ in symbols (Mackey, 1963)] if there exist three pairwise orthogonal propositions a_1, b_1, and c such that $a = a_1 \vee c$ and $b = b_1 \vee c$. This relation is obviously symmetric and reflexive. The phenomenological meaning of the so-defined relation $\leftrightarrow$ is not clear, but it can easily be understood if we notice the following (Varadarajan, 1962):

$a \leftrightarrow b$ iff there exist an observable $x: B(R^1) \to L$ and Borel sets $E, F \in B(R^1)$ such that $a = x(E)$ and $b = x(F)$

Hence, as a direct consequence, we obtain

$$a \leftrightarrow b \quad \text{implies} \quad a \leftrightarrow b'$$

The following statement may help clarify the significance of the relation $\leftrightarrow$ [for its proof, see, e.g., Varadarajan (1962)]:

$a \leftrightarrow b$ ($a, b \in L$) iff there exists a Boolean sublogic of L containing both a and b.

The following statements can be shown (Varadarajan, 1962):

(1) Let $a, b \in L$ and $a \leftrightarrow b$, that is

$$a = a_1 \vee c \quad \text{and} \quad b = b_1 \vee c$$

where a_1, b_1, and c are pairwise orthogonal. Then there exist $a \vee b$ and $a \wedge b$, and $a \wedge b = c$.

(2) Let $a, a_1, a_2, \ldots \in L$. If $a \leftrightarrow a_i$ for all $i = 1, 2, \ldots$ and if $\bigvee_{i=1}^{\infty} a_i$ and $\bigvee_{i=1}^{\infty} (a \wedge a_i)$ both exist, then $a \leftrightarrow \bigvee_{i=1}^{\infty} a_i$ and

$$a \wedge \left(\bigvee_{i=1}^{\infty} a_i\right) = \bigvee_{i=1}^{\infty} (a \wedge a_i)$$

The following properties of the relation $\leftrightarrow$ are of particular importance to us (Varadarajan, 1962):

(a) $\leftrightarrow$ is symmetric and reflexive.
(b) $a \leftrightarrow b$ implies $a \leftrightarrow b'$.
(c) $a \leqslant b$ implies $a \leftrightarrow b$.
(d) $a \leftrightarrow b$, $a \leftrightarrow c$, $b \perp c$ imply $a \leftrightarrow b \vee c$.

Note that (c) is a direct consequence of the orthomodularity of L, and (d) follows from (2).

Definition. Every binary relation $C \subseteq L \times L$ satisfying the preceding conditions, i.e., such that

(i) C is symmetric and reflexive,
(ii) $a\, C\, b$ implies $a\, C\, b'$,
(iii) $a \leqslant b$ implies $a\, C\, b$,
(iv) $a\, C\, b$, $a\, C\, c$, $b \perp c$ imply $a\, C\, b \vee c$,

will be called the *weak compatibility* in L.

The justification for such a name for C is contained in the following statement.

Theorem 1. The relation $\leftrightarrow$ is the strongest one in the family of all relations C with properties (i)–(iv), that is

$$a \leftrightarrow b \qquad \text{implies} \qquad a\, C\, b$$

In other words, $\leftrightarrow \subseteq C$ for any relation C satisfying the conditions (i)–(iv).

Proof. Suppose that $a \leftrightarrow b$ $(a, b \in L)$. Then $a = a_1 \vee c$ and $b = b_1 \vee c$, where a_1, b_1, and c are mutually orthogonal. Hence $b_1 \perp a_1 \vee c = a$, and so $b_1\, C\, a$ by (iii) and (ii). But $c \leqslant a$ implies $c\, C\, a$ by (iii), and therefore by (iv) and (i) one finds $a\, C\, b_1 \vee c = b$. This proves that $\leftrightarrow \subseteq C$ indeed.

Corollary. Let $\mathscr{C}$ denote the family of all weak compatibilities in L. Then

$$\leftrightarrow = \bigcap_{C \in \mathscr{C}} C$$

Definition. If the weak compatibility $C \subseteq L \times L$ has the additional property

$$a\, C\, b \Rightarrow \text{there exists } a \vee b \text{ in } L$$

(or the equivalent dual property $a\, C\, b \Rightarrow$ there exists $a \wedge b$ in L), and if instead of (iv) the relation C satisfies the stronger condition

(iv′) $a\, C\, b, a\, C\, c, b\, C\, c \Rightarrow a\, C\, b \vee c$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

or the equivalent dual condition, then we shall say that C is *regular*.

It is well known (Pool, 1963; Ramsay, 1966) that not every logic admits a regular weak compatibility on it. On the other hand, in any lattice logic the relation $\leftrightarrow$ is regular. This follows easily from (1) and (2). So the existence of a regular weak compatibility in the logic L may be considered a "regularity" condition for L, and thus the following definition can be accepted.

Definition. The logic L is said to be *regular* if a regular weak compatibility may be defined in L.

> *Theorem 2.* If a weak compatibility $C \subseteq L \times L$ is regular, then $C = \leftrightarrow$. Thus there exists at most one regular weak compatibility in L.

The proof of this theorem is preceded by two lemmas.

> *Lemma 1.* Let C be a regular weak compatibility in L, and let M be a nonempty subset of L. Then M is a Boolean lattice, whenever the following conditions are satisfied:
>
> (a) For any pair $a, b \in M$ one has $a\,C\,b$ and $a \vee b \in M$.
> (b) $a \in M$ implies $a' \in M$.

Proof. From (a) and (b) it follows readily that $a \wedge b \in M$ for all $a, b \in M$ and that 0 and 1 are in M. Thus M becomes an orthocomplemented lattice, and it remains to be shown that M is distributive. But the distributivity of M is guaranteed by the property (iv′) of the relation C. Indeed, owing to (a) and (iv′) we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in M$, which, by (b), implies the dual identity $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, all $a, b, c \in M$.

> *Lemma 2.* Let C be a regular weak compatibility in the logic L. Then every subset of L consisting of pairwise weakly compatible propositions is contained in some Boolean sublattice of L.

Proof. Consider the family $\mathscr{A}$ of all subsets $A \subseteq L$ consisting of mutually weakly compatible elements of L (that is, such subsets $A \subseteq L$ for which $a\,C\,b$ for any pair $a, b \in A$), partially ordered by the set inclusion. If $\{A_t\}_{t\in T}$ is an arbitrary chain of elements from $\mathscr{A}$, then obviously $\bigcup_{t\in T} A_t$ belongs also to $\mathscr{A}$. Of course, $\bigcup_{t\in T} A_t$ is an upper bound for the chain $\{A_t\}_{t\in T}$; hence by Zorn's lemma every subset $A \in \mathscr{A}$ is contained in some maximal set $M \in \mathscr{A}$ for which

(i) $a\,C\,b$ for all $a, b \in M$ (since $M \in \mathscr{A}$).
(ii) $a \in M$ implies $a' \in M$ (by the maximality of M).
(iii) $a \vee b \in M$ for all $a, b \in M$ (by the maximality of M and the regularity of C).

All conditions of Lemma 1 are satisfied, and hence M is a Boolean sublattice of L.

The proof of the theorem is now immediate: $a\,C\,b$ implies $a, b \in M$ for some Boolean sublattice $M \subseteq L$ by Lemma 2; hence [see, e.g., Varadarajan (1962)] $a \leftrightarrow b$. We thus have shown that $C \subseteq \leftrightarrow$, which together with the inverse inclusion proved in Theorem 1 leads to $C = \leftrightarrow$, as claimed.

Thus the regularity of an arbitrary weak compatibility $C \subseteq L \times L$ implies the regularity of the relation $\leftrightarrow$.

Note, by the way, the following fact:

> *Theorem 3.* The relation $\leftrightarrow$ is regular if and only if it satisfies the first part of condition (iv′), that is, if the following property holds for $\leftrightarrow$:
>
> (*) For any triple a, b, c of mutually compatible propositions one has $a \leftrightarrow b \vee c$.

Proof. One needs to show only the "if" part of the theorem. Suppose thus that condition (*) is satisfied by $\leftrightarrow$. Hence, by the condition dual to (*) [which is, of course, equivalent to (*)], we get

$$a \leftrightarrow b,\ a \leftrightarrow c,\ b \leftrightarrow c \Rightarrow c \leftrightarrow a \wedge b$$

Hence

$$a \wedge b \leftrightarrow a \wedge c$$

since $a \wedge b \leqslant a$ leads to $a \wedge b \leftrightarrow a$ [see, e.g., Guz (1971)], and therefore we conclude [see (1)] that there exists $(a \wedge b) \vee (a \wedge c)$.

Since $a \leftrightarrow b \vee c$ by (*), there exists, by (1), $a \wedge (b \vee c)$. It now remains to appeal to (2) in order to find that

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

which is the second part of condition (iv′). The proof of the theorem is thus complete.

A direct consequence of Theorem 3 is the following result.

> *Corollary.* The logic L is regular if and only if the compatibility relation possesses the property (*) or the equivalent property dual to (*).

Examples, which have been found by Pool (1963) and independently by Ramsay (1966), show that there are logics in which the condition (*) is not satisfied. In other words, not every logic is regular. We shall now formulate another condition, being necessary and sufficient for the logic L to be regular. It is interesting because of its connection with the concept of commensurability of observables associated with the logic.

Definition. We say that two observables $x, y: B(R^1) \to L$ are *commensurable* (or *simultaneously measurable*), and write $x \leftrightarrow y$, if there exist an observable z and Borel functions $f, g: R^1 \to R^1$ such that $x = f(z)$ and $y = g(z)$.

Remark. By $f(x)$, where x is an arbitrary observable associated with L, is meant the observable defined by

$$(f(x))(E) = x(f^{-1}(E))$$

It can be shown [see, for example, Ramsay (1966)] that

$$x \leftrightarrow y \qquad \text{iff} \qquad x(E) \leftrightarrow y(F) \text{ for all } E, F \in B(R^1)$$

Usually just the preceding condition was taken as defining the commensurability of observables x and y. However, from the physical point of view our former definition seems to be more plausible.

The following important theorem was stated by Varadarajan (1962):

(∗∗) For every sequence $\{x_n\}_{n=1}^{\infty}$ of pairwise commensurable observables there exist an observable x and Borel functions f_n: $R^1 \to R^1$ such that $x_n = f_n(x)$, $n = 1, 2, \ldots$.

However, Varadarajan's proof of (∗∗) was not correct because he implicitly assumed property (∗) for the compatibility relation $\leftrightarrow$; as we know [see counterexamples given by Pool (1963) and Ramsay (1966)] this assumption does not hold in an arbitrary logic. Thus the question arises: Is the condition (∗) necessary for the validity of the theorem (∗∗)? Alternatively, is there another proof of (∗∗) that does not use property (∗)? The answer to this question is given by the following theorem (Guz, 1971).

For any logic L the following two conditions are equivalent:

(i) L is regular.

(ii) The theorem (∗∗) holds in L.

This theorem gives us the necessary and sufficient condition for the regularity of L. However, because of the complexity of condition (∗∗), it seems more appropriate to take the regularity of the logic L as the necessary and sufficient condition for the validity of Varadarajan's theorem (∗∗) in L.

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